# Chapter 1
# Expressions and Equations

**Abstract** In this section we will introduce you to the very foundations of algebra. We will start by showing how the natural numbers initiate the evolution of the number systems, which stretches all the way to complex numbers and will go on to show how these number systems are linked to one another. We will then present some of the basic algebraic expressions that you would encounter throughout this book and their geometric rationale. We will then, gently, introduce the art of solving equations by presenting the simplest scenario which involves the solving of single and multiple linear equations and inequalities using different approaches.

## 1.1 Introduction

In school's mathematics curricula, Algebra has gone through major changes in the past few years. The point at which students should learn algebra and how they should learn it has been debated throughout the United States, particularly in California, where people promote taking algebra in earlier grades. However, the concept that algebra is a subject that is a gateway for all other mathematical scientific content needs justification.

Those intending to teach in secondary school mathematics classrooms as well as those intending to teach in elementary school classrooms can benefit from improving their knowledge of algebra. The need for deeper conceptual understanding is essential for those teachers.

## 1.2 Number Systems

### *1.2.1 Types of Numbers and Notation*

In mathematics, numbers are commonly categorized in various sets, some of these number sets are subsets of other number sets. Alphabetical letters such as $\mathbb{N}$, $\mathbb{W}$, $\mathbb{Z}$, $\mathbb{R}$, and $\mathbb{Q}$ are commonly used to represent these number sets. Understanding how these numbers are related to each other and how they are categorized is essential to learning the concepts explained in this book.

$\mathbb{N}$: The set of all natural numbers.

$$\mathbb{N} = \{1,2,3,4,\cdots\}$$

These are also called counting numbers. These are the first numbers humans used for counting. Note that 0 is not included in the natural numbers.

$\mathbb{W}$: The set of whole numbers.

$$\mathbb{W} = \{0,1,2,3,4,\cdots\}$$

The whole numbers are composed of zero and the set of natural numbers. Humans started using zero much later in their development.

$\mathbb{Z}$: The set of all integers.

$$\mathbb{Z} = \{\cdots,-4,-3,-2,-1,0,1,2,3,4,\cdots\}$$

The integers are composed of whole numbers and their opposites. Often people separate the integers into positive integers (natural numbers), negative integers (opposites of natural numbers), and zero.

$\mathbb{Q}$: The set of all rational numbers.

$$\mathbb{Q} = \left\{ \frac{a}{b};\ a, b \text{ are integers},\ b \neq 0 \right\}$$

Rational numbers are numbers that can be expressed as a ratio of two integers. All integers are rational numbers. Another common definition of rational numbers is: a set of numbers that, when written in decimal form, either terminate or repeat.

Numbers that cannot be written as a ratio of two integers are called Irrational Numbers. $\sqrt{2}$, $\sqrt{3}$, and $\pi$ are some of the most common irrational numbers.

$\mathbb{R}$: The set of all real numbers.

$$\mathbb{R} = \{a;\ a \text{ is a rational number or irrational number}\}$$

Any point on the number line (often referred to as the real line) has a corresponding real number. Numbers that are not defined as real numbers, are called imaginary numbers. The square root of a negative number is an imaginary number. $\sqrt{-1}$ is represented by the letter $i$.

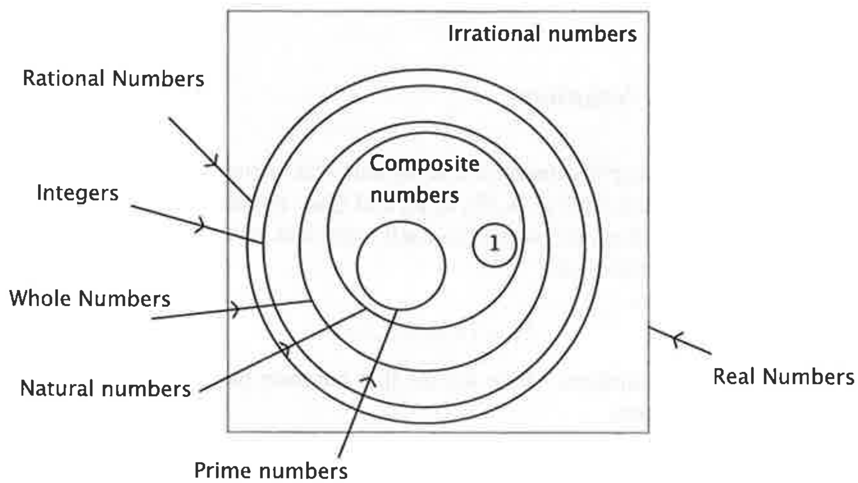
$\mathbb{C}$: The set of all complex numbers.

$$\mathbb{C} = \{a + bi;\ a \text{ and } b \text{ are real numbers}\}$$

Complex numbers consist of real numbers, imaginary numbers and their combinations.

### *1.2.2 How Various Numbers are Related to Each Other*

Many of the sets of numbers are subsets of the other sets. For example, the set of rational numbers and set of irrational numbers are subsets of the set of real numbers. Also, the set of natural numbers, set of whole numbers and the set of integers are subsets of the set of rational numbers. The following Venn diagram is a visual representation of how these various sets of numbers relate to each other.




## 1.3 Algebraic Expressions

Real numbers, variables and combinations of them with mathematical operations are called algebraic expressions. For example, the following are algebraic expressions:

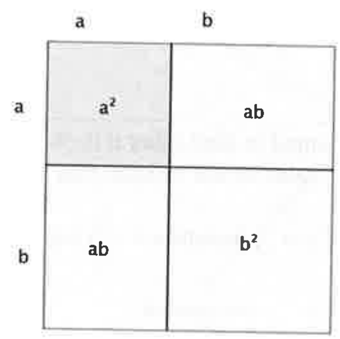
$$-5, \ 100, \ \frac{3}{5}, \ 2x+3, \ 6x+y, \ -7xy+3, \ \frac{2x}{3x+1}$$

Algebraic expressions can be evaluated if they have variables. For example, the expression $3x+4$ evaluated for $x=2$ is $3(2)+4=10$. Some algebraic expressions may be simplified by combining like terms, factoring or distributing.

Manipulatives such as Algebra Tiles can be used to visualize algebraic expressions. These tiles allow us to learn to simplify and evaluate expressions, as long as one does not forget to consider the order of operations when working with them. Showing how to do this is beyond the scope of this book, however all readers are encouraged to explore and investigate further to learn and teach manipulations of expressions using Algebra Tiles. You can learn about virtual manipulatives at `http://nlvm.usu.edu`.

The area model is used for multiplication of whole numbers and fractions since it helps the learner to visualize the distributive property easily. The same model can be also used to study more complex algebraic expressions. In the following example we show how to obtain a very common and important 'formula' using the area model.

**Example 1.1.** Consider the figure




We look at the area of this square in two different ways and we obtain.

$$(a+b)^2 = (a+b)(a+b) = a^2+2ab+b^2 \qquad \boxed{(a+b)^2 = a^2+2ab+b^2}$$

Next you will find a list of very common, and important, algebraic expressions that you will encounter in middle and high school mathematics curriculum. Familiarizing yourself with how to simplify and expand them will be very useful in learning algebra. It is a good exercise for you to create the appropriate figure to explain how these expressions are obtained.

$$\begin{aligned}(a-b)^2 &= (a-b)(a-b)\\ &= a^2-2ab+b^2\end{aligned} \qquad \boxed{(a-b)^2 = a^2-2ab+b^2}$$

$$\begin{aligned}(a-b)(a+b) &= a^2-ab+ab-b^2\\ &= a^2-b^2\end{aligned} \qquad \boxed{(a-b)(a+b) = a^2-b^2}$$

$$\begin{aligned}(x-a)(x-b) &= x^2-ax-bx+ab\\ &= x^2-(a+b)x+ab\end{aligned} \qquad \boxed{(x-a)(x-b) = x^2-(a+b)x+ab}$$

$$\begin{aligned}(a+b)^3 &= (a+b)^2(a+b)\\ &= (a^2+2ab+b^2)(a+b)\\ &= a^3+3a^2b+3ab^2+b^3\end{aligned} \qquad \boxed{(a+b)^3 = a^3+3a^2b+3ab^2+b^3}$$

$$\begin{aligned}(a-b)^3 &= (a-b)^2(a-b)\\ &= (a^2-2ab+b^2)(a-b)\\ &= a^3-3a^2b+3ab^2-b^3\end{aligned}$$

$$\boxed{(a-b)^3 = a^3-3a^2b+3ab^2-b^3}$$

$$\begin{aligned}(a+b)(a^2-ab+b^2) &= a(a^2-ab+b^2)+b(a^2-ab+b^2)\\ &= a^3-a^2b+ab^2+a^2b-ab^2+b^3\\ &= a^3+b^3\end{aligned}$$

$$\boxed{(a+b)(a^2-ab+b^2) = a^3+b^3}$$

$$\begin{aligned}(a-b)(a^2+ab+b^2) &= a(a^2+ab+b^2)-b(a^2+ab+b^2)\\ &= a^3+a^2b+ab^2-(a^2b+ab^2+b^3)\\ &= a^3+a^2b+ab^2-a^2b-ab^2-b^3\\ &= a^3-b^3\end{aligned}$$

$$\boxed{(a-b)(a^2+ab+b^2) = a^3-b^3}$$

## 1.4 Equations

When you set two algebraic expressions equal to each other it is called an equation. If you have one variable in the equation, you may be able to find the value of the variable that satisfies the equation. This process is called *solving the equation.*

For example $2x+4=10$ is true when $x=3$, therefore $x=3$ satisfies the equation.

### *1.4.1 Solving Linear Equations*

A linear equation in one variable is an equation of the form of $ax+b=0$, where $a$ and $b$ are real numbers, with $a \neq 0$.

Why is it called linear? Because the solutions of an equation of the form $y=ax+b$ form a line.

When you have one linear equation with one variable you can find one unique solution. You can do this by solving algebraically or solving graphically. However, when you have one linear equation with two variables you can find an infinite number of pairs of values that satisfy that equation.

When you have two equations with two variables it is called system of equations. For a system of equations you have the possibility of having one solution, no solution or infinitely many solutions. Just like when having one linear equation with two variables, you can solve system of equations graphically or algebraically. Solving algebraically can be done using many techniques, such as substitution, elimination and by using matrices (you will learn this in later chapter in this book).

#### Solving Algebraically

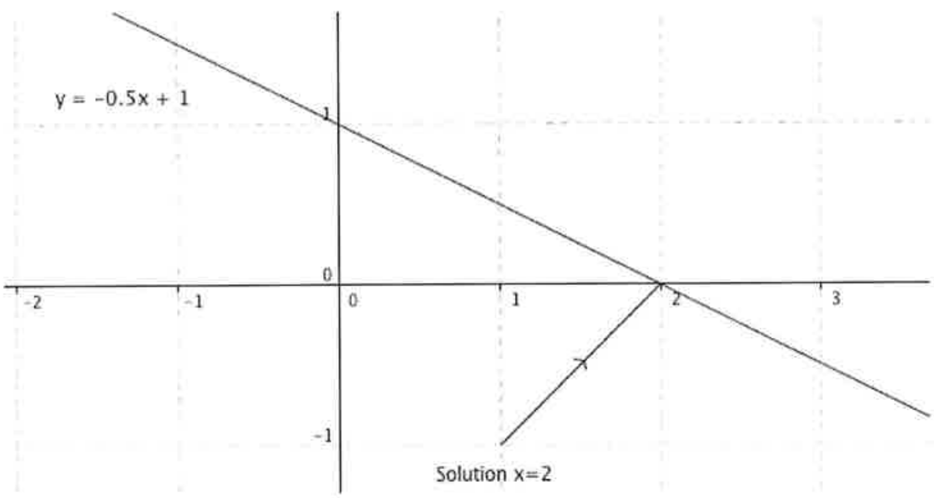
Solving linear equations can be done using many strategies. To solve an equation algebraically is to find the value of the variable by isolating it on one side of the equation. For this, we use the addition property of equality and the multiplication property of equality.

<u>Addition Property of Equality:</u> Adding the same number to both sides of an equation does not change the solution set to the equation. In symbols: if $a=b$, then $a+c=b+c$.

Multiplication Property of Equality: Multiplying both sides of an equation by the same non-zero number does not change the solution set to the equation. In symbols: if $a = b$, then $ac = bc$, when $c \neq 0$.

**Solving Graphically**

To find the solution to $ax + b = 0$, you can graph the equation $y = ax + b$ and find the place where this graph intersects the $x$-axis (when $y = 0$). Because $y = ax + b$ represents a straight line on the plane, then the set of all pairs $(x, y)$ on the line are the solutions to $y = ax + b$. So, if you want to solve an equation, such as $-\frac{x}{2} + 1 = 0$ you need to graph $y = -\frac{x}{2} + 1$ and find its $x$-intercept. The following figure illustrates this idea.




## 1.5 Solving a System of Equations

Any collection of two or more equations is called a system of equations. The set of all values of the variables that satisfy all equations is called the solution set of the system. In this section, we will look at various strategies to solve systems of two linear equations with two variables.

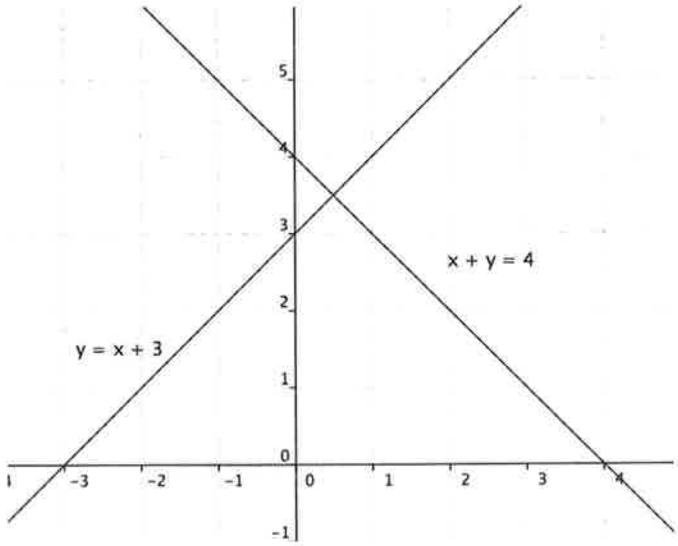
### *1.5.1 Solving a System by Graphing*

We plot the two lines and look for the points where the lines intersect (if they do). This method is not so effective, as most of the times it will be very hard to find what the intersection points is.

The three possible situations are represented next:

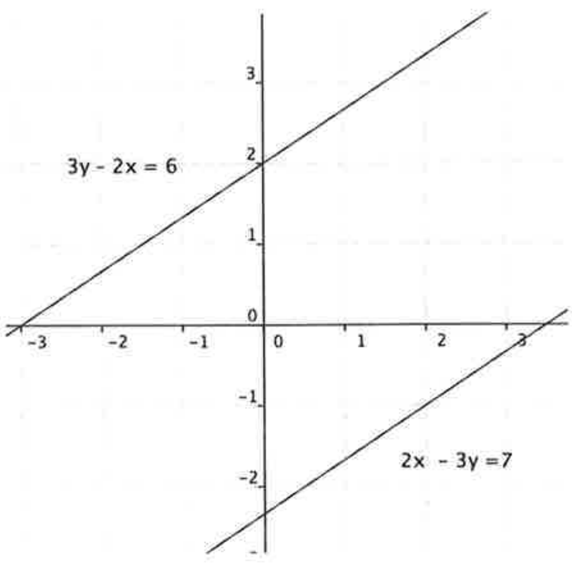
A system with one solution:

$$y = x + 3$$
$$x + y = 4$$



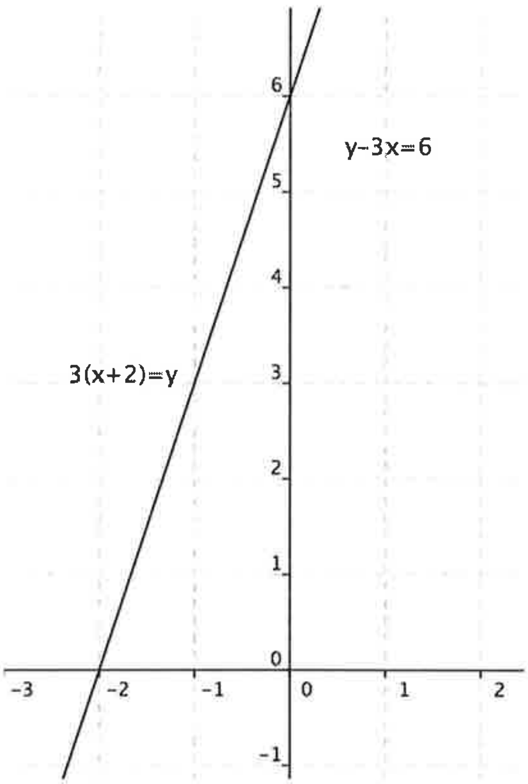

A system with no solutions:

$$2x - 3y = 7$$
$$3y - 2x = 6$$



A system with infinitely many solutions:

$$3(x+2) = y$$
$$y - 3x = 6$$




## 1.6 Solving Linear Inequalities

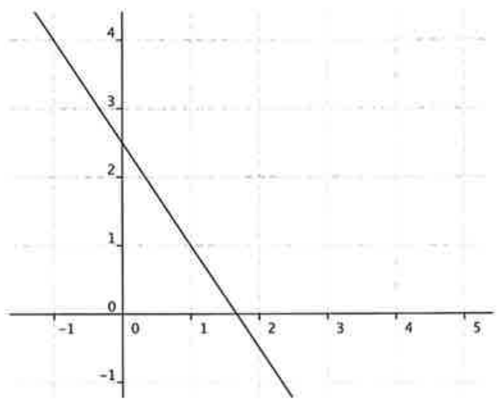
In this section, we will discuss how to graph linear inequalities of two variables in the $X-Y$-plane. Also, we will discuss how to find a region in the $X-Y$-plane common to all of them (called the feasible region) when you have several linear inequalities (called a system of inequalities).
*What is a linear inequality ?*
It is similar to a linear equality, but instead of the "=" sign, you have one of the following: "$>$", "$<$" , "$\geq$", "$\leq$".
So, for example $3x+2y \leq 5$ is an example of a linear inequality.

## 1.7 The Graph of a Single Linear Inequality

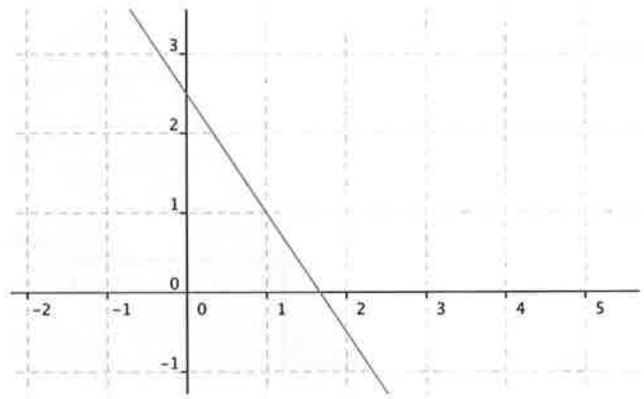
Now what does a linear inequality represent in the $X-Y$-plane?
Linear inequalities represent **half planes** in the $X-Y$-plane.
As an illustration,

(i) First let us graphically illustrate the **linear equality** $3x+2y = 5$ in the $X-Y$-plane.
You can easily do this by finding the $x$-intercept and the $y$-intercept.

$X$-intercept: put $y = 0$. We get $3x+2\cdot 0 = 5 \to 3x = 5 \to x = 5/3$ .
$Y$-intercept: put $x = 0$. We get $3\cdot 0+2y = 5 \to 2y = 5 \to y = 5/2$.
Now let us us mark the $x$ and the $y$-intercepts (5/3 and 2.5) and draw the graph.




(ii) Now let us graphically illustrate the **linear inequality** $3x+2y \leq 5$




Notice that the region shaded in blue is the graph of the inequality. This is a half plane. Can you now see what the boundary of the half plane is? Notice that the boundary of the half plane represented by the **inequality** $3x+2y \leq 5$ is the **equality** $3x+2y = 5$. So, now you can see a way to actually graph an inequality
**How to graph an inequality:**

Step 1. Change the 'unequal' sign to an 'equal' sign and get the equality.
Step 2. Draw the straight line corresponding to the equality (easiest way is to find two points like the $x$ and $y$-intercepts by setting $y = 0$ and $x = 0$, respectively, and then draw the line using the $x$ and $y$-intercepts) This straight line is the boundary of the half plane that we want.
But the line boundary has two sides! How do we know which side is correct?
Step 3. To find the correct side, we select one side of the line at random, and select a test point that lies on that side (usually we can select $(0,0)$, unless that point is actually on the border line). Now you plug the coordinates of the test point into the inequality and check whether the inequality is true for that test point. If the inequality is true, then the test point is lying on the correct side and you shade that side. Otherwise, the test point is on the wrong side. In this case, the correct side is the other side and you shade that other side.

**The Graph of a Single Linear Inequality: Worked out examples**

1. *Find the region in the $X-Y$-plane represented by the inequality $y+x \leq 1$*
**Answer:** Let us go through each of the steps one by one.
**Step 1: Change the 'unequal' sign to an 'equal' sign and get an equality.**

Inequality: $y+x \leq 1$

Changing the sign to an 'equal' sign, we get the equality: $y+x=1$

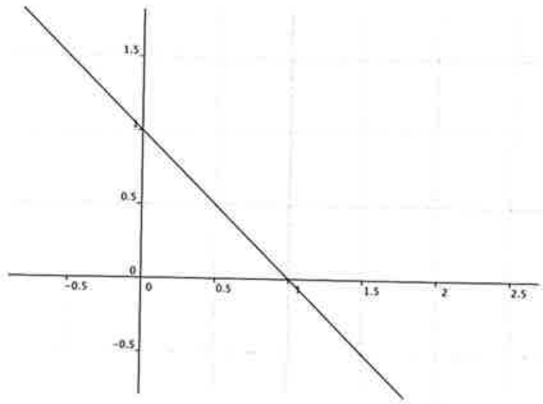
**Step 2: Draw the straight line corresponding to the equality.**

Now we will draw the straight line representing the equality: $y+x=1$

**Find $x$-intercept:** When $y=0$, since $y+x=1$, then $x=1$ is the $x$-intercept.

**Find $y$-intercept:** When $x=0$, since $y+x=1$, then $y=1$ is the $y$-intercept

Now, using $x$ and $y$-intercepts, we can draw the straight line $y+x=1$ corresponding to the equality



Now that we have finished drawing the equality straight line, we need to find which side of the line (left or right) satisfies the inequality. To do that, we go to step 3.

**Step 3: Identify the the correct side to shade.**

Let's check whether left side is the correct side. Notice that the point $(0,0)$ is on the left side. Let us plug it into the inequality $y+x \leq 1$.

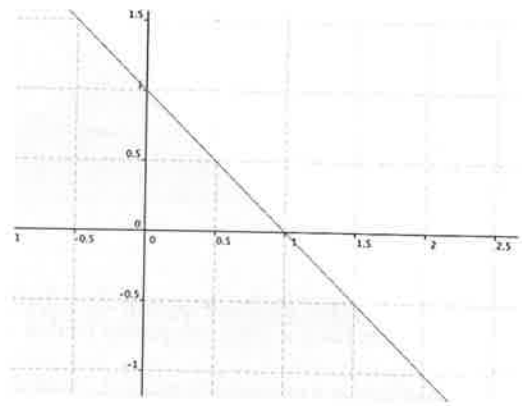
Plugging in $x=0$ and $y=0$, we get $0+0 \leq 1$. Is this a true statement?

Yes, since $0 \leq 1$.

So, the side that we chose is the correct side !

The left side of the line represents the inequality.

Now let us shade this region, a half plane, as the answer



The area shaded in blue is the area that we require.

2. *Find the region in the $X-Y$ plane represented by the inequality* $2x-y \leq -2$.

**Answer:** Let us go through each of the steps one by one

**Step 1: Change the 'unequal' sign to an 'equal' sign and get an equality.**

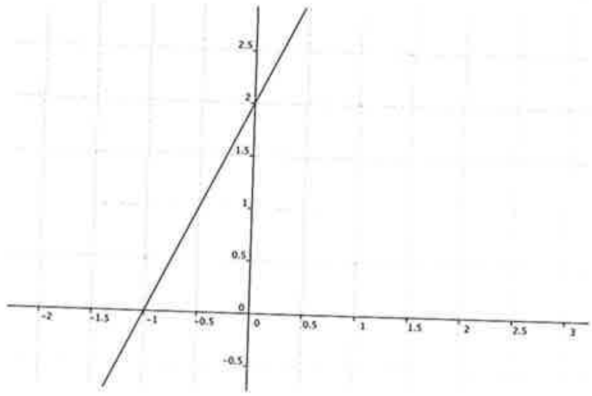
Inequality: $2x-y \leq -2$

Changing the sign to an 'equal' sign we get the equality: $2x-y=-2$

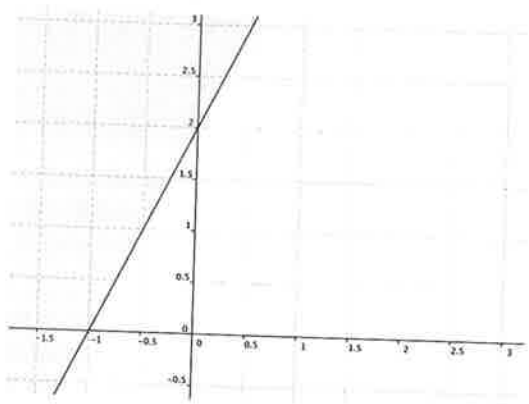
**Step 2: Draw the straight line corresponding to the equality.**

Now we will draw the straight line representing the equality: $2x-y=-2$

**Find $x$-intercept:** When $y = 0$, since $2x - y = -2$, then $x = -1$ is the $x$-intercept.
**Find $y$-intercept:** When $x = 0$, since $2x - y = -2$, then $y = 2$ is the $y$-intercept.
Now using $X$ and $Y$ intercepts we can draw the straight line $2x - y = -2$ corresponding to the equality



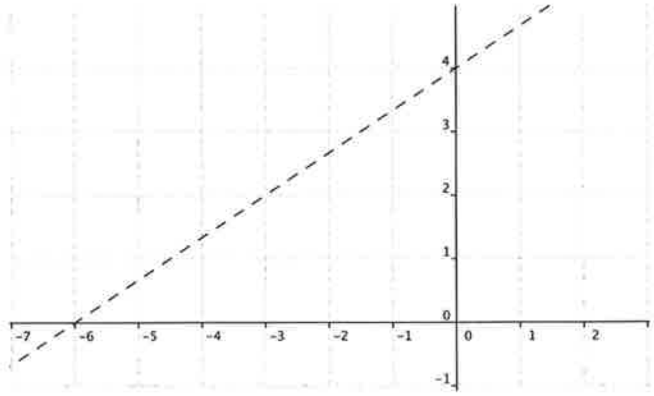

Now we need to find which side of the line is the correct side for the inequality.
**Step 3: Identify the the correct side to shade.**
Let us check whether left side is the correct side. Notice that the point $(0,0)$ is on the left side. Let us plug it to the inequality $2x - y \leq -2$.
Plugging in $x = 0$ and $y = 0$, we get $0 + 0 \leq -2$. Is this a true statement ? No, since 0 cannot be less than $-2$.
This means only one thing $(0,0)$ **is not on the correct side.** Then what is the correct side?
**The other side, the side not containing $(0,0)$, is the correct side.**
Now let us shade this side. This is the answer.



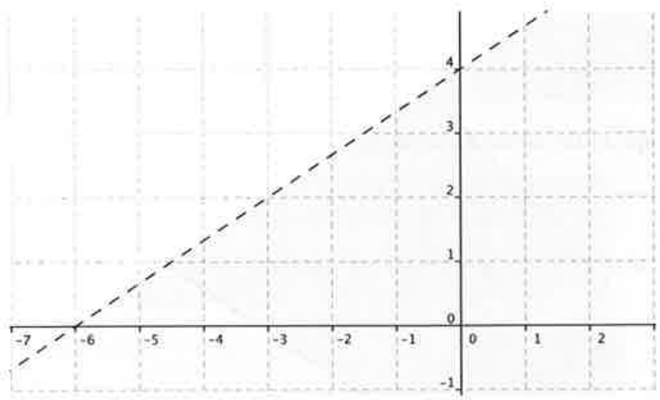

The area shaded is the area that we require.

3. *Find the region in the $X - Y$-plane represented by the inequality* $2x - 3y > -12$.
**Answer:** Notice that in this problem we have a strict inequality (rather than 'greater than or equal...' we have 'strictly greater than...').
The only difference between this problem and the previous problem is that when you have a strict inequality, the boundary straight line is a dotted line, meaning you don't consider the points on the straight line as part of the region.
Let us go through each of the steps one by one
**Step 1: Change the 'unequal' sign to an 'equal' sign and get an equality.**
Inequality: $2x - 3y > -12$
Changing the sign to an 'equal' sign we get the equality: $2x - 3y = -12$
**Step 2: Draw the straight line corresponding to the equality.**
Now we will draw the straight line representing the equality: $2x - 3y = -12$
**Find $x$-intercept:** When $y = 0$, since $2x - 3y = -12$, then $x = -6$ is the $x$-intercept.

**Find $y$-intercept:** When $x = 0$, since $2x - 3y = -12$, then $y = 4$ is the $y$-intercept
Now using $X$ and $Y$ intercepts we can draw the straight line $2x - 3y = -12$ corresponding to the equality.




Notice however since we have the strict inequality, then we must use a dotted line when drawing the line for the final answer.
Now we need to find which side of the line is the correct side for the inequality.
**Step 3: Identify the the correct side to shade.**
As before, consider $(0,0)$ as a test point and plug it in.
Plugging in $x = 0$ and $y = 0$, we get $0 + 0 > -12$. Is this a true statement ? Yes, since 0 is greater than any negative number.
$(0,0)$ **is on the correct side !**
So, the correct side is the side of the straight line which contains $(0,0)$. Now let us shade the final answer.




## 1.8 Graphing a System of Two or More Inequalities

In the earlier examples you learned how to draw the graph of a single inequality, which we saw represents a half plane. Now let us try to graph two inequalities in the same $X - Y$-plane.
For example, we would like to graph the system

$$\begin{cases} x + 2y \leq 8 \\ 2x - 3y \leq 9 \end{cases}$$

Note that, by graphing these two inequalities, what we intend to do is to find points $(x, y)$ that simultaneously satisfy both these inequalities. How do we do we do this?

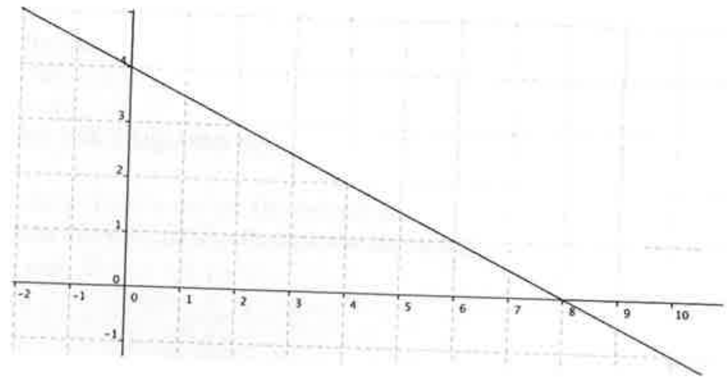
First we find the region representing one of the inequalities using the method we used in the earlier exercises. Then we find the region representing the second inequality. Then we look to see whether there is a common region. That common region is the graph of the system of inequalities. We will try this in the following examples.

**Graphing a System of Two or More Inequalities: Worked out examples**

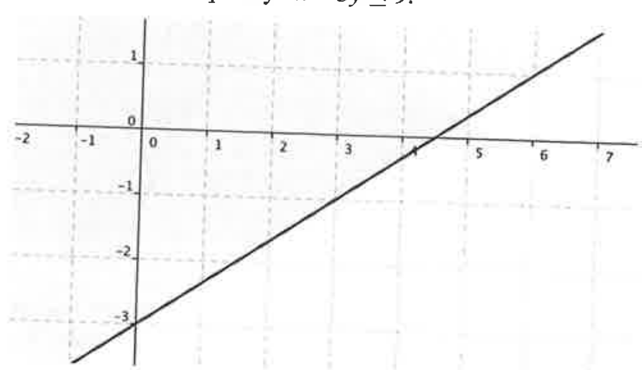
1. *Graph the system*

$$\begin{cases} x+2y \leq 8 \\ 2x-3y \leq 9 \end{cases}$$

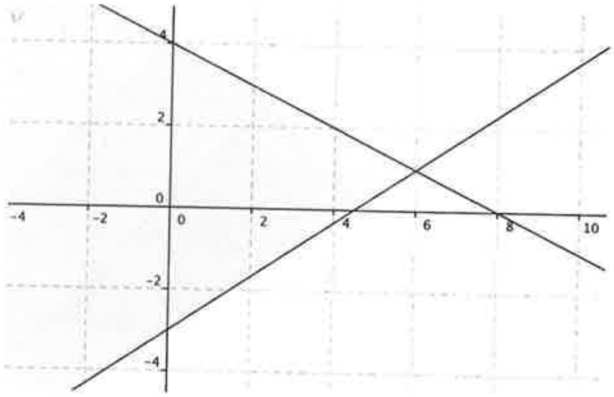
**Answer:** We will first draw the graph of the first inequality $x+2y \leq 8$.
As before first consider $x+2y=8$.
***x*-intercept:** Input $y=0$. We get $x=8$.
***y*-intercept:** Input $x=0$. We get $y=4$.
Use the test point $(0,0)$. We get $0<8$, which is a true statement. Therefore $(0,0)$ is on the correct side of the inequality. We get the following graph.




Next we draw the graph of the second inequality $2x-3y \leq 9$.



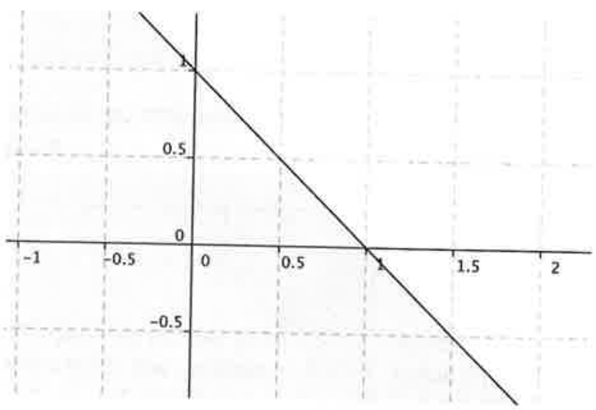

When you draw these two regions in the same diagram, you will notice that there is a region that is common to both these regions. This common region consists of the points $(x,y)$ that simultaneously satisfy both these inequalities. That common region is our solution, and it is called *the feasible region of the system.*
The region common to both of the above regions (solution of the system) is shown below.



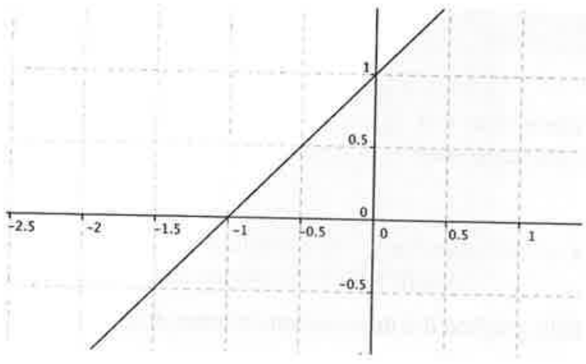

2. *Graph the system of inequalities*

$$\begin{cases} x+y \leq 1 \\ y-x \leq 1 \end{cases}$$

**Answer:** First we should graph the inequality $x+y \leq 1$, but this was already done in problem 1 of section 1.7. We got the following graph:

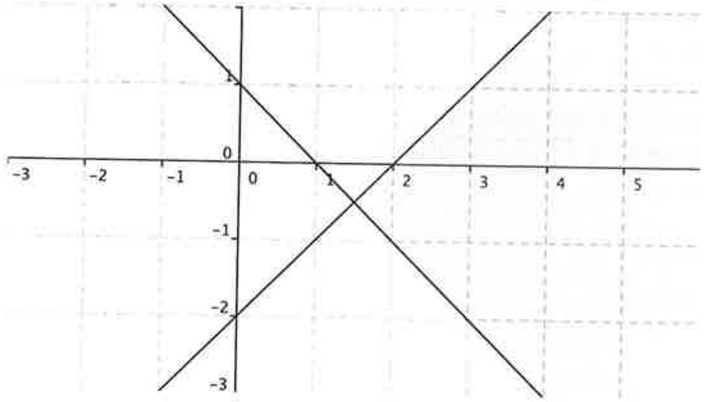



Next we graph the inequality $y-x \leq 1$ by graphing the line $y-x=1$ and then plugging $(0,0)$ into the inequality to decide which side to shade. We get:




So far we have individually graphed the two regions represented by the two inequalities. Now we simply graph them in the same graph and find the common region:

7. *Find a system of two inequalities that has a solution region of the shaded region below*




**Answer:** We have already solved a problem of this type (problem 5 above). Hence, we will proceed in the same way we did then.
The first order of business is to find the equations of the straight lines that form the boundary.

(a) What is the equation of the line with the positive slope?
Notice that the points $(2,0)$ and $(0,-2)$ are on this line. Using this we can find the slope of the line

$$m = \frac{-2-0}{0-2} = 1$$

and then using the point-slope formula we get $y = x-2$.

(b) Similarly, the equation of the line with negative slope can be calculated and it is $y = -x+1$.
In order to find the inequalities, we choose a point in the shaded region. For instance, take $(3,0)$ and plug it into

$$y \,\square\, x-2 \qquad\qquad y \,\square\, -x+1$$

to get

$$0 < 3-2 \qquad\qquad 0 > -3+1$$

The system we were looking for is

$$\begin{cases} y \leq x-2 \\ y \geq -x+1 \end{cases} \qquad i.e. \qquad \begin{cases} y-x \leq -2 \\ y+x \geq 1 \end{cases}$$

## Exercises

**1.1.** Factor the following expressions
(i) $ax^2 - 16$
(ii) $a^3 - 27$
(iii) $b^2 + 6b + 8$
(iv) $x^2 - x - 6$
(v) $20x^2 + 47x + 21$

**1.2.** Graph (solve) the following inequalities.

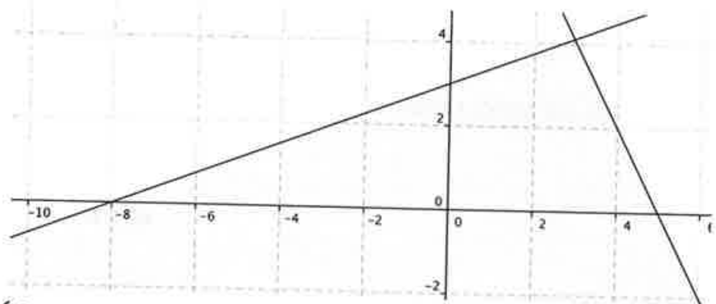
(*i*) $\begin{cases} y-x\geq -1 \\ y+2x\leq 4 \end{cases}$ (*ii*) $\begin{cases} 2x-y\leq 8 \\ x-3y\geq 6 \end{cases}$ (*iii*) $\begin{cases} 2x+y>5 \\ 5x-3y<15 \end{cases}$

(*iv*) $\begin{cases} 2x-y>-3 \\ 4x+y<5 \end{cases}$ (*v*) $\begin{cases} 3x+2y\geq 15 \\ x\geq 3 \end{cases}$ (*vi*) $\begin{cases} 2x-3y\leq 12 \\ x+5y\leq 20 \\ x>0 \end{cases}$

(*vii*) $\begin{cases} x+3y\geq 12 \\ 3x+2y\leq 15 \\ y\geq 2 \end{cases}$ (*viii*) $\begin{cases} 2x+3y\geq 6 \\ y-x\geq 0 \\ y\leq 2 \end{cases}$ (*ix*) $\begin{cases} 4x+2y\geq 28 \\ x\geq 0 \\ y\geq 0 \\ 2x+y\geq 12 \\ 5x+8y\geq 74 \end{cases}$

(*x*) $\begin{cases} x+6y\geq 24 \\ x\geq 0 \\ y\geq 0 \end{cases}$

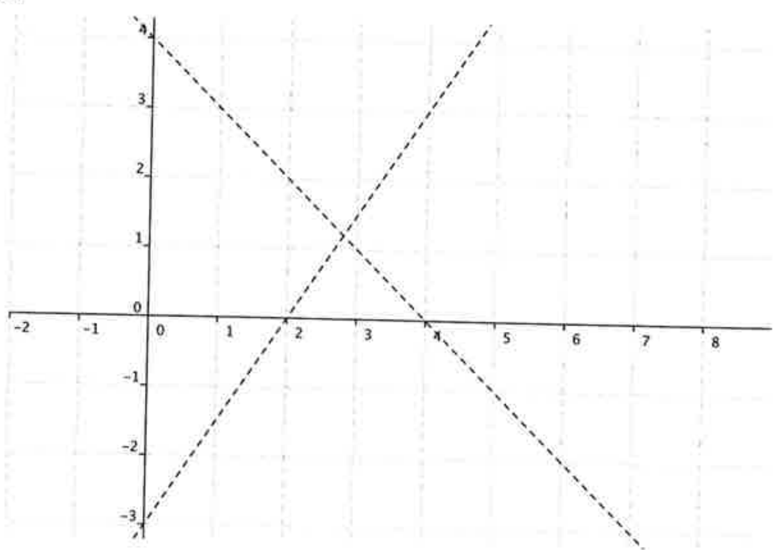
**1.3.** Which of the following inequalities represent the region shaded below




(*a*) $\begin{cases} 8y-3x\geq 24 \\ y+2x\leq 10 \end{cases}$ (*b*) $\begin{cases} 8y-3x\leq 24 \\ y+2x\geq 10 \end{cases}$

(*c*) $\begin{cases} 3x-8y\geq -24 \\ y+2x\leq 10 \end{cases}$ (*d*) $\begin{cases} 8y-3x\geq 24 \\ y+2x\geq 10 \end{cases}$

**1.4.** Write four systems of inequalities, one for each of the four regions in the $X-Y$-plane formed by the following two lines.



# Chapter 2
# Quadratic Equations

**Abstract** In this chapter we will discuss solving quadratic equations using both *completing the square* and the quadratic formula. Applications of the quadratic formula appear in SMR 1.2 of the CSET subtest 1 examination.

A quadratic equation is an equation of the form $ax^2 + bx + c = 0$, where $a \neq 0$ and $a, b$, and $c$ could be real or complex numbers. There are many ways to solve such an equation, and there are many problems that can be solved by using quadratic equations. In this chapter you will learn, among other things, the following topics regarding quadratic equations

1. How to solve a given quadratic equation using the quadratic formula.
2. How to solve a quadratic equation using completing the square.
3. How to solve a rational equation, which leads to a quadratic equation.
4. How to solve an equation, which can be transformed into quadratic equations.
5. How to handle a quadratic equation with complex coefficients
6. How to manipulate the roots of a quadratic equation without solving the equation.

## 2.1 The Quadratic Formula

The quadratic formula is the name we give to the formula that says that the solutions of the equation $ax^2 + bx + c = 0$, where $a \neq 0$, are given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

The number $\Delta = b^2 - 4ac$ is called the discriminant of the equation $ax^2 + bx + c = 0$. Since computing $\sqrt{\Delta}$ yields either two distinct real numbers (when $\Delta > 0$), or one real number (which is zero, when $\Delta = 0$), or two distinct complex (no real) numbers (when $\Delta < 0$) then

- if $\Delta > 0$, then the equation has exactly two distinct real solutions,
- if $\Delta = 0$, then the equation has exactly one real solution,
- if $\Delta < 0$, then the equation has exactly two distinct complex (non-real) solutions.

If you want to use the quadratic formula to solve a given quadratic equation you can use a two-step approach:
Step 1. Write the given equation in the standard form: $ax^2 + bx + c = 0$.
Step 2. Identify what $a$, $b$ and $c$ are, and the plug these values in the quadratic formula to get the solution(s).

### The Quadratic Formula: Worked out examples

1. *Solve* $x^2 - 5x + 6 = 0$.
   **Answer:** Since this equation is already in standard form, we can immediately go to step 1 and find $a, b$ and $c$. In this case we get $a = 1$, $b = -5$ and $c = 6$.
   Now we plug these values into the quadratic formula and obtain

$$x = \frac{-(-5) \pm \sqrt{(-5)^2 - 4(1)(6)}}{2(1)} = \frac{5 \pm \sqrt{25 - 24}}{2} = \frac{5 \pm 1}{2}$$

We get the solutions to be $x = \frac{5-1}{2} = 2$ and $x = \frac{5+1}{2} = 3$.
Notice also that in this case the discriminant $\Delta = 1 > 0$, which means the equation has two distinct real roots. This is consistent with what we have found above.

2. *Solve* $x^2 = 2 - 4x$.
**Answer:** This equation is not yet in standard form, so we need to put it in the standard, which is $x^2 + 4x - 2 = 0$.
Now we look for the values to plug into the quadratic formula, and we get $a = 1$, $b = 4$ and $c = -2$. By plugging these values we obtain

$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(1)(-2)}}{2(1)} = \frac{-4 \pm \sqrt{16+8}}{2} = \frac{-4 \pm \sqrt{24}}{2} = \frac{-4 \pm 2\sqrt{6}}{2} = -2 \pm \sqrt{6}$$

So, the solutions are $x = -2 - \sqrt{6}$ and $x = -2 + \sqrt{6}$.

3. *Solve* $x^2 + 8x + 16 = 0$.
**Answer:** This equation is already in standard form, so we just identify $a = 1$, $b = 8$ and $c = 16$. We plug these values into the quadratic formula to obtain

$$x = \frac{-(8) \pm \sqrt{(8)^2 - 4(1)(16)}}{2(1)} = \frac{-8 \pm \sqrt{64-64}}{2} = \frac{-8 \pm \sqrt{0}}{2} = -4$$

So, there is only one solution, which is $x = -4$. Note that this is consistent with having $\Delta = 0$ for this equation.

4. *Solve* $x^2 - 2x + 2 = 0$.
**Answer:** Since this equation is already in standard form, then we start by identifying $a = 1$, $b = -2$ and $c = 2$. We plug these values into the quadratic formula to obtain

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(2)}}{2(1)} = \frac{2 \pm \sqrt{4-8}}{2} = \frac{2 \pm \sqrt{-4}}{2} = \frac{2 \pm 2\sqrt{-1}}{2} = 1 \pm \sqrt{-1}$$

Since $\sqrt{-1} = i$, the we get two complex roots: $x = 1 - i$ and $x = 1 + i$. Note that this is consistent with having $\Delta = -4 < 0$ for this equation.

5. *Solve the equation* $(x+2)(x+1) = 12$.
**Answer:** Even though this equation does not seem to be a quadratic equation, it is. In order to see this one just needs to multiply the expression on the left-hand side. Let us do this

$$12 = (x+2)(x+1) = x^2 + 3x + 2$$

and thus, the equation we want to solve is $x^2 + 3x - 10 = 0$.
Now we just plug $a = 1$, $b = 3$ and $c = -10$ into the quadratic formula to obtain

$$x = \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(-10)}}{2(1)} = \frac{-3 \pm \sqrt{9+40}}{2} = \frac{-3 \pm \sqrt{49}}{2} = \frac{-3 \pm 7}{2}$$

So, we get the two solutions to be $x = -5$ and $x = 2$.

## 2.2 Rational Equations That Lead to Quadratic Equations

Consider the equation

$$8 - 4x = \frac{1}{x}$$

At first sight, this does not look like a quadratic equation. But if you multiply both sides of the equation by $x$ then you do get the following quadratic equation:

$$8x - 4x^2 = 1$$

So, in order to solve a rational equation one should try to multiply both sides of the equation by an expression that eliminates all denominators (multiplying by the *LCM* of the denominators is a good idea). After multiplying and simplifying you might end up with a quadratic equation, which you can solve using steps 1 and 2 that we outlined in the previous section.

**Rational Equations That Lead to Quadratic Equations: Worked out examples**

1. *Solve* $8 - 4x = \frac{1}{x}$.
   **Answer:** As we have seen above, after multiplying by $x$ both sides of this equation we get $8x - 4x^2 = 1$, which once re-written as $8x - 4x^2 - 1 = 0$ we can solve by setting $a = 8$, $b = -4$, and $c = -1$ and plugging these values in the quadratic formula to get

$$x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(8)(-1)}}{2(8)} = \frac{4 \pm \sqrt{48}}{16} = \frac{4 \pm 4\sqrt{3}}{16} = \frac{1 \pm \sqrt{3}}{4}$$

   Therefore the solutions are $x = \frac{1-\sqrt{3}}{4}$ and $x = \frac{1+\sqrt{3}}{4}$.
2. *Solve* $\frac{x-2}{x-3} = x + 2$.
   **Answer:** In this case, we need to multiply both sides by $x - 3$ to eliminate all denominators. Once we do this we get

$$x - 2 = (x+2)(x-3)$$

   which can be re-written by multiplying the terms on the right-hand side to get $x - 2 = x^2 - x - 6$, which after simplifying becomes, $x^2 - 2x - 4 = 0$
   Since in this case we have $a = 1$, $b = -2$, and $c = -4$ then

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)} = \frac{2 \pm \sqrt{20}}{2} = \frac{2 \pm 2\sqrt{5}}{2} = 1 \pm \sqrt{5}$$

   So, the solutions are $x = 1 - \sqrt{5}$ and $x = 1 + \sqrt{5}$.

## 2.3 Quadratic Equations With Complex Coefficients

As mentioned at the beginning of this chapter, the coefficients $a, b$ and $c$ of the equation $ax^2 + bx + c = 0$ could be real or complex. So far we have solved only equations with real coefficients, but in this section we will discuss equations that might have complex coefficients. In short, there is no need to worry about this 'strange' numbers, you should proceed as if you were solving a quadratic equation with real coefficients.

**Example 2.1.** We want to solve $z^2 + (2 - i)z - i = 0$.
Comparing with the standard quadratic equation $ax^2 + bx + c = 0$ (of course now we have the variable $z$ instead of $x$) we see that $a = 1$, $b = 2 - i$, and $c = -i$ then

$$\begin{aligned} z &= \frac{-(2-i) \pm \sqrt{(2-i)^2 - 4(1)(-i)}}{2(1)} \\ &= \frac{-2+i \pm \sqrt{2^2 - 2\cdot 2i + (i^2) + 4i}}{2} \\ &= \frac{-2+i \pm \sqrt{3}}{2} \end{aligned}$$

So, we have two solutions for the equation: $z = \dfrac{-2-\sqrt{3}+i}{2}$ and $z = \dfrac{-2+\sqrt{3}+i}{2}$.

### Quadratic Equations With Complex Coefficients: Worked out examples

1. *Find the imaginary part of the solutions of* $2iz^2 + (4+i)z + 1 = 0$.
   **Answer:** In order to find the imaginary part of the solutions of the equation, we first need to solve the equation. It is clear that $a = 2i$, $b = 4+i$, and $c = 1$ then

$$\begin{aligned} z &= \frac{-(4+i) \pm \sqrt{(4+i)^2 - 4(2i)(1)}}{2(2i)} \\ &= \frac{-4-i \pm \sqrt{(16+8i-1)-8i}}{4i} \\ &= \frac{-4-i \pm \sqrt{15}}{4i} \end{aligned}$$

   Now to find the imaginary parts, first we notice that $i$ appears both in the numerator and the denominator. To get rid of the $4i$ in the denominator we will multiply both the numerator and the denominator by $-4i$. Hence,

$$\begin{aligned} z &= \frac{(-4-i \pm \sqrt{15})(-4i)}{-(4i)^2} \\ &= \frac{16i + 4i^2 \pm \sqrt{15}\,4i}{16} \\ &= \frac{-4 + (16 \pm 4\sqrt{15})i}{16} \\ &= \frac{-4}{16} + \frac{16 \pm 4\sqrt{15}}{16} i \end{aligned}$$

   It follows, after simplifying, that the imaginary parts of the roots are $\dfrac{4 \pm \sqrt{15}}{4}$.

2. *Find the solutions, and their imaginary parts, for* $z^2 + 2z + 4i - 2 = 0$.
   **Answer:** It is clear that $a = 1$, $b = 2$, and $c = 4i - 2$. Thus, the quadratic formula yields

$$\begin{aligned} z &= \frac{-(2) \pm \sqrt{(2)^2 - 4(1)(4i-2)}}{2(1)} \\ &= \frac{-2 \pm \sqrt{4(3-4i)}}{2} \\ &= -1 \pm \sqrt{3-4i} \end{aligned}$$

   In any one of the previous problems we have ever had a non-real number under the the radical sign. What do we do with that $3-4i$ under the radical? Well, our best hope is that $3-4i$ is a square, in which case the

radical would cancel with the square and then we would have a clean expression, with no square roots. Let us see if we can get $3-4i$ to be square.
Since $(s+ti)^2=(s^2-t^2)+2sti$ then we want to find $s$ and $t$ such that $3=s^2-t^2$ and $-4=2st$. We see that $s=2$ and $t=-1$ solve these equations, and thus $(2-i)^2=3-4i$. Hence, the solutions of the equation are

$$z=-1\pm\sqrt{3-4i}=-1\pm\sqrt{(2-i)^2}=-1\pm(2-i)$$

which, explicitly, are $x=-3+i$ and $x=1-i$.

## 2.4 Equations That Transform Into Quadratic Equations

Consider the problem of solving the equation $x^4-5x^2+4=0$, which is an equation of degree 4.
Note that by using the simple substitution $t=x^2$ we can transform this equation into a quadratic equation. In fact, we get

$$t^2-5t+4=0$$

which has solutions $t=1$ and $t=4$. Hence, the solutions for the original equation are given by $x^2=1$ and $x^2=4$. This implies that the solutions are $x=\pm1$ and $x=\pm2$.

**Equations That Transform Into Quadratic Equations: Worked out examples**

1. *Solve* $x^4-13x^2+36=0$.
   **Answer:** Using the substitution $t=x^2$ we obtain the equation $t^2-13t+36=0$, which we can solve using the quadratic formula, we get
   $$t=\frac{13\pm\sqrt{(-13)^2-4\cdot36}}{2}=\frac{13\pm5}{2}$$
   It follows that the solutions for this equation are $t=4$ and $t=9$. So, we now know that $x^2=4$ and $x^2=9$, and thus $x=\pm2$ and $x=\pm3$.
2. *Solve* $x^{2/3}-3x^{1/3}+2=0$.
   **Answer:** In this equation setting $t=x^2$ will lead us nowhere, but since $x^{2/3}=(x^{1/3})^2$ then we can consider the substitution $t=x^{1/3}$ to get the equation.
   $$t^2-3t+2=0$$
   Now we use the quadratic formula to find the solutions to this equation. We get $t=1$ and $t=2$. It follows that $x^{1/3}=1$, and thus $x=1$ or $x^{1/3}=2$, and thus $x=8$.
   Since this is a radical equation it is of vital importance to check whether the solutions found are correct. We check:
   For $x=1$
   $$1^{2/3}-3\cdot1^{1/3}+2=1-3+2=0$$
   For $x=8$
   $$8^{2/3}-3\cdot8^{1/3}+2=4-3\cdot2+2=4-6+2=0$$
   So, both $x=1$ and $x=8$ are valid solutions.
3. *Solve* $a-6\sqrt{a}+8=0$.
   **Answer:** This is just like the previous example except that in this case we will use you the substitution $t=\sqrt{a}$, which will transform the given equation into $t^2-6t+8=0$, which can be solved using the quadratic formula. The solutions of this equation are $t=2$ and $t=4$. Now we back-substitute to get $\sqrt{a}=2$ and $\sqrt{a}=4$ implying $a=4$ and $a=16$.
   It is easy to check that both solutions make sense and are valid.

## 2.5 Manipulation of Roots of Quadratic Equations

Sometimes it becomes necessary for us to get information about the roots of a quadratic equation without actually solving the equation. In order to do this we will use a couple of very simple formulas.

Let $\alpha$ and $\beta$ be the two solutions (not necessarily distinct) of the equation $ax^2+bx+c=0$, then:

$$(i)\ \alpha+\beta=-\frac{b}{a} \qquad (ii)\ \alpha\beta=\frac{c}{a} \qquad (iii)\ |\alpha-\beta|=\frac{\sqrt{\Delta}}{a}$$

where $\Delta=b^2-4ac$, the discriminant of the equation.

Why does all this make sense? Note that if $\alpha$ and $\beta$ are the two solutions of $ax^2+bx+c=0$ then

$$ax^2+bx+c=a(x-\alpha)(x-\beta)$$

and thus, when multiplying the expression on the right we get

$$ax^2+bx+c=ax^2-a(\alpha+\beta)x+a\alpha\beta$$

By comparing the coefficients with $x$ and no $x$ we get the $\alpha+\beta=-\frac{b}{a}$ and $\alpha\beta=\frac{c}{a}$.

The expression $|\alpha-\beta|=\frac{\sqrt{\Delta}}{a}$ follows from taking the difference of the solutions given by the quadratic formula, that is

$$|\alpha-\beta|=\left|\frac{-b+\sqrt{b^2-4ac}}{2a}-\frac{-b-\sqrt{b^2-4ac}}{2a}\right|=\left|\pm\frac{\sqrt{b^2-4ac}}{a}\right|=\frac{\sqrt{b^2-4ac}}{a}$$

**Manipulation of Roots of Quadratic Equations: Worked out examples**

1. *Find the sum and the product of the roots of the equation* $4x^2-8x+3=0$.
   **Answer:** This can be calculated easily by simply plugging values in the formulae given.

$$\alpha+\beta=-\frac{b}{a}=-\frac{-8}{4}=2 \qquad \alpha\beta=\frac{c}{a}=\frac{3}{4}$$

2. *Find the absolute value of the difference of the roots of the equation* $4x^2-16x+15=0$.
   **Answer:** Then again, from the formulae given above, we can calculated this easily

$$|\alpha-\beta|=\frac{\sqrt{(16)^2-4\cdot4\cdot15}}{4}=\frac{\sqrt{16}}{4}=1$$

## 2.6 Solving Quadratic Equations by Completing the Square.

Using the quadratic formula is one way of solving a quadratic equation. Another way is to use a method called *completion of a square*. We will describe this method in the following steps.

Assume the quadratic equation has the form $ax^2+bx+c=0$.

Step 1: Subtract the constant $c$ from both sides so that only the $x^2$ and $x$ terms are left on the left-hand side.

Step 2: Make the coefficient of $x^2$ equal to 1 by dividing both sides by the coefficient $a$. Note that now the coefficient of $x$ has also been modified.

Step 3: Add $\left(\frac{1}{2}\cdot\text{new coefficient with } x\right)^2$ both sides.

Step 4: Write the left hand side as a complete square. You should have

$$\left(x - \frac{1}{2} \cdot \text{new coefficient with } x\right)^2$$

Step 5: Take square roots on both sides. Make sure to put $\pm$ on front of the radical.
Step 6: Solve for $x$.

**Solving Quadratic Equations by Completing the Square: Worked out examples**

1. *Solve* $x^2+6x+5=0$ *by completing the square.*
   **Answer:** Let us perform the steps described above.
   Step 1: Subtract 5 both sides: $x^2+6x=-5$
   Step 2: Since the coefficient with $x^2$ is 1, there is nothing to do.
   Step 3: 1/2 of the coefficient with $x$ is 3, then we add $3^2=9$ both sides: $x^2+6x+9=4$
   Step 4: The left-hand side is a perfect square: $(x+3)^2=4$.
   Step 5: We take square roots both sides: $x+3=\pm 2$.
   Step 6: $x=-3\pm 2$, and thus $x=-1$ or $x=-5$.
2. *Solve* $2x^2+6x-3=0$ *by completing the square.*
   **Answer:** Just as in the previous problem, we will perform the steps described above.
   Step 1: Subtract $-3$ both sides: $2x^2+6x=3$
   Step 2: The coefficient with $x^2$ is 2, then we divide by 2: $x^2+3x=\frac{3}{2}$
   Step 3: 1/2 of the coefficient with $x$ is $\frac{3}{2}$, then we add $\left(\frac{3}{2}\right)^2$ both sides, we get

   $$x^2+3x+\left(\frac{3}{2}\right)^2=\frac{3}{2}+\left(\frac{3}{2}\right)^2$$

   Step 4: The left-hand side is a perfect square, and we simplify the right-hand side:

   $$\left(x+\frac{3}{2}\right)^2=\frac{15}{4}$$

   Step 5: We take square roots both sides: $x+\frac{3}{2}=\pm\sqrt{\frac{15}{4}}$.
   Step 6: $x=\pm\sqrt{\frac{15}{4}}-\frac{3}{2}=\frac{-3\pm\sqrt{15}}{2}$
3. *Solve* $x(2x-1)=-\frac{7}{2}$ *by completing the square.*

   **Answer:** We first distribute and get the equation in the standard form: $2x^2-x+\frac{7}{2}=0$. Now we can use the steps.
   Step 1: Subtract $\frac{7}{2}$ both sides: $2x^2-x=-\frac{7}{2}$
   Step 2: The coefficient with $x^2$ is 2, then we divide by 2: $x^2-\frac{1}{2}x=-\frac{7}{4}$.
   Step 3: 1/2 of the coefficient with $x$ is $\frac{1}{4}$, then we add $\left(\frac{1}{4}\right)^2$ both sides, we get

   $$x^2-\frac{1}{2}x+\left(\frac{1}{4}\right)^2=-\frac{7}{4}+\left(\frac{1}{4}\right)^2$$

   Step 4: The left-hand side is a perfect square, and we simplify the right-hand side:

$$\left(x-\frac{1}{4}\right)^2=-\frac{27}{16}$$

Step 5: We take square roots both sides: $x-\frac{1}{4}=\pm\sqrt{-\frac{27}{16}}$.

Step 6: $x=\frac{1}{4}\pm\sqrt{-\frac{27}{16}}=\frac{1\pm3\sqrt{-3}}{4}=\frac{1\pm3\sqrt{3}i}{4}$

## Exercises

**2.1.** Fill in the following table

| Equation | Discriminant $\Delta=b^2-4ac$ | Types of Roots |
|---|---|---|
| $x^2-9x+18=0$ | | |
| $x^2-10x+25=0$ | | |
| $x^2+2x+7=0$ | | |
| $\frac{x^2}{2}+\frac{3x}{5}=\frac{3}{10}$ | | |
| $\frac{x}{2}+\frac{2}{3}=\frac{x^2}{6}$ | | |

**2.2.** Solve the equations:
(i) $2x^2-12=-5x$.
(ii) $x^2-3x=-1$.
(iii) $y^2+3y=8$.
(iv) $(x-2)(x+3)=24$.
(v) $2x^2-2x=-7$.

**2.3.** Solve the equations:
(i) $4x+5=\frac{6}{x}$
(ii) $8x-8=\frac{3}{x}$
(iii) $1+\frac{9}{2x}=\frac{5}{2x^2}$

**2.4.** Solve the equations:
(i) $y^4-5y^2+4=0$.
(ii) $x^4-8x^2+16=0$.
(iii) $2x^{2/3}-7x^{1/3}+6=0$.
(iv) $x^{2/5}-x^{1/5}-6=0$.
(v) $(x-2)^2-8(x-2)+15=0$.
(vi) $a+27=12\sqrt{a}$.
(vii) $y-36=9\sqrt{y}$.
(viii) $9\left(\frac{1}{x+2}\right)^2-10\left(\frac{1}{x+2}\right)=-1$.
(ix) $a^3(a^3-7)=8$.

**2.5.** Find the solutions of
(i) $z^2 - 2iz - 3 = 0$.
(ii) $iz^2 + (1 - 5i)z - 1 + 8i = 0$.

**2.6.** Find the imaginary parts of the solutions of $iz^2 + 3z - 2i = 0$.

**2.7.** What are the real and imaginary parts of the solutions of the equation

$$(1+i)z^2 + (3-2i)z - (21-7i) = 0\ ?$$

**2.8.** If $\alpha$ and $\beta$ are the solutions of the equation $x^2 - 2x - 3 = 0$. Find

$(i)\ \alpha\beta \qquad (ii)\alpha + \beta \qquad (iii)|\alpha - \beta| \qquad (iv)\alpha^2 + \beta^2$

**2.9.** Find the absolute value of the difference of the roots of the equation $2x^2 - x - 3 = 0$.

**2.10.** You are given that the absolute value of the difference of the roots of $x^2 - mx + 1 = 0$ is equal to $\sqrt{5}$, where $m$ is a real positive number. Find $m$.

**2.11.** Solve the following quadratic equations by completing squares
(i) $x^2 + 4x = 9$.
(ii) $2x^2 - 4x - 1 = 0$.
(iii) $6x = 1 - 4x^2$.
(iv) $x(2x+9) = -5$.